The Astronomical Journal, 136:1030–1038, 2008 September
doi:10.1088/0004-6256/136/3/1030

N-PARAMETRIC CANONICAL PERTURBATION METHOD BASED ON LIE TRANSFORMS

M. Andrade[1]
Astronomical Observatory R. M. Aller, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Galiza, Spain
Received 2007 July 3; accepted 2008 June 13; published 2008 August 1

ABSTRACT

In the dynamical analysis of multiple stellar systems, one usually has to deal with problems involving several perturbations: more than two bodies, mass loss, nonspherical shape, rotation, and such. In order to take into account all of them, we have derived a multiparametric theory based on Lie transforms. It allows us to solve perturbation problems involving an arbitrary number of small parameters in the Hamiltonian formulation. Based on the Lie transforms theory, a complete generalization of the Hori–Deprit method is obtained for N parameters—with N arbitrary—and general expressions are explicitly provided. This method is used to solve the classical Gyldén–Meščerskij problem—the relative motion of a binary system the components of which are losing mass over time—when the primary's oblateness, as well as relativistic effects, is taken into account. Besides this, speed and accuracy comparisons between this analytical method and a numerical one are accomplished.

Key words: binaries: general – celestial mechanics – methods: analytical – relativity – stars: mass loss – stellar dynamics

1. INTRODUCTION

The need for solving nonlinear differential equations arises in the study of many dynamical problems. Since, in general, the exact solution cannot be found by means of classical integration methods, it is necessary to use alternative analytical or numerical methods. The perturbation methods based on asymptotic developments of the equations of motion in terms of one or more small parameters belong to the first class.

In the present paper we deal with perturbation theories for nonlinear dynamical systems based on the so-called method of *Lie transforms*. The first methods of this type, referring to the case of Hamiltonian systems depending on one small parameter, were given by Hori (1966) and Deprit (1969). Although these methods are equivalent (Campbell & Jefferys 1970; Henrard & Roels 1974), they are not identical. Later, Kamel (1970) and Henrard (1970) generalized them to arbitrary systems of differential equations. Ribera (1981) and Abad & Ribera (1984) obtained a Lie transform method applicable to Hamiltonian systems depending on two small parameters. This was used by Prieto & Docobo (1997a, 1997b) to analytically integrate the two-body problem with slowly decreasing mass. Varadi (1985) also obtained a two-parameter method based on differential geometry, as in Henrard & Roels (1974). This was subsequently extended to the case of three parameters by Ahmed (1993). Recently, a three-parameter canonical method (Andrade 2002) has been applied to the integration of the two-body problem with mass loss depending both on the time and the distance of the bodies (Docobo & Andrade 2002).

Despite these partial results, there are a considerable number of interesting problems in different fields in which it is necessary to consider a large number of small parameters and their associated perturbative expansions. To our knowledge, no generalization of the Lie canonical method has so far been presented in the case of N parameters, with N arbitrary. This is precisely the purpose of the present paper.

In particular, we provide below (Section 2) the general formulae of a canonical method based on Lie transforms, for an undefined number of small parameters. To illustrate this method, we apply it (Section 3) in a four-parametric case, namely the Gyldén–Meščerskij problem of relative motion of two stars that are losing mass over time when, in addition, the primary's oblateness, as well as relativistic effects, is taken into account. Conclusions are summarized in Section 4.

2. MATHEMATICAL DERIVATIONS

2.1. *Transformation of a Hamiltonian by Means of an N-parametric Group of Canonical Transformations*

Let a dynamical system of n degrees of freedom be defined by the flow under the Hamiltonian $\mathcal{H} = \mathcal{H}(\vec{x}, t; \varepsilon)$, where $\vec{x} = (\mathbf{x}, \mathbf{X}) \in \mathbb{R}^n \times \mathbb{R}^n$, $\mathbf{x}$, $\mathbf{X}$ are canonical coordinates and momenta, respectively. The associated canonical system that describes the dynamical evolution of the system is

$$\dot{\vec{x}} = J\,\nabla_{\vec{x}}\mathcal{H},$$

where J is the symplectic matrix ($J^{-1} = J^T = -J$)

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix},$$

with 0_n and I_n being the null and unity matrix of nth order, respectively.

Let us suppose that the Hamiltonian can be developed as a power series of an N-dimensional parameter $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$ belonging to a neighborhood U_0 of the origin $(0, 0, \ldots, 0)$ of $\mathbb{R}^N$, with $N \in \mathbb{Z}^+$:

$$\mathcal{H}(\vec{x}, t; \varepsilon) = \mathcal{H}_0(\vec{x}, t) + \sum_{j_N \geqslant 1} \frac{1}{j_N!} \sum_{j_{N-1}=0}^{j_N} \ldots \sum_{j_2=0}^{j_3} \sum_{j_1=0}^{j_2} \varepsilon_1^{j_1} \varepsilon_2^{j_2-j_1} \times \ldots \varepsilon_N^{j_N-j_{N-1}} \mathcal{H}_{j_1, j_2-j_1, \ldots, j_N-j_{N-1}}(\vec{x}, t), \tag{1}$$

where

$$\mathcal{H}_{j_1, j_2-j_1, \ldots, j_N-j_{N-1}}(\vec{x}, t) \equiv \binom{j_N}{j_{N-1}} \ldots \binom{j_2}{j_1} \frac{\partial_1^{j}}{\partial \varepsilon_1^{j_1}} \frac{\partial^{j_2-j_1}}{\partial \varepsilon_2^{j_2-j_1}} \times \ldots \frac{\partial^{j_N-j_{N-1}}}{\partial \varepsilon_N^{j_N-j_{N-1}}} \mathcal{H}(\vec{x}, t; 0).$$

[1] Departamento de Matemática Aplicada, Escola Politécnica Superior, Universidade de Santiago de Compostela, Campus de Lugo, 27002 Lugo, Galiza, Spain.

Now, consider an N-parametric Lie group of canonical transformations G_N of the extended phase space $\mathbb{R}^{2n+N}$ given by

$$\vec{x} = e^{Y_\varepsilon}\vec{y},$$

where the Y_ε operator is defined by means of the Poisson brackets

$$Y_\varepsilon = \{-; \mathcal{W}_\varepsilon\},$$

with $\mathcal{W}_\varepsilon(\vec{y}, t)$ as the generating function of the transformation. This can be expressed as power series

$$\mathcal{W}_\varepsilon(\vec{y}, t; \varepsilon) = \sum_{i=1}^{N} \varepsilon_i \mathcal{W}^i(\vec{y}, t; \varepsilon_i), \tag{2}$$

where

$$\mathcal{W}^i(\vec{y}, t; \varepsilon_i) = \sum_{j\geqslant 0} \frac{\varepsilon_i^j}{j!}\mathcal{W}^i_{j+1}(\vec{y}, t), \qquad i = 1, \ldots, N.$$

The canonical property of the above transformation can be demonstrated by considering that the generating function itself is a sum of N independent parts, so that each of them is a solution of the Hamilton equations of a dynamical system of Hamiltonian $\mathcal{W}^i$ and time ε_i. Furthermore, we can write Y_ε as

$$Y_\varepsilon = \sum_{i=1}^{N} \varepsilon_i Y_i,$$

with $Y_i = \{-; \mathcal{W}^i\}$, $i = 1, \ldots, N$.

Proposition. *The transformed Hamiltonian of (1) is given by*

$$\mathcal{H}^*(\vec{y}, t; \varepsilon) = \mathcal{H}_0(\vec{y}, t) + \sum_{j_N\geqslant 1} \frac{Y_\varepsilon^{j_N}}{j_N!}\mathcal{H}_0(\vec{y}, t) + \sum_{j_N\geqslant 1}\left[\frac{1}{j_N!}\sum_{j_{N-1}=0}^{j_N} \times \ldots \sum_{j_2=0}^{j_3}\sum_{j_1=0}^{j_2} \varepsilon_1^{j_1}\varepsilon_2^{j_2-j_1}\ldots\varepsilon_N^{j_N-j_{N-1}} \cdot\left[\mathcal{H}_{j_1, j_2-j_1, \ldots, j_N-j_{N-1}}(\vec{y}, t) + \sum_{i\geqslant 1}\frac{Y_\varepsilon^i}{i!}\mathcal{H}_{j_1, j_2-j_1, \ldots, j_N-j_{N-1}}(\vec{y}, t)\right]\right]. \tag{3}$$

Proof. Expansion (1) can be cast in the form

$$\mathcal{H}(\vec{x}, t; \varepsilon) = \mathcal{H}_0(\vec{x}, t) + \sum_{j_N\geqslant 1} F_{j_N}(\vec{x}, t; \varepsilon), \tag{4}$$

where

$$F_{j_N}(\vec{x}, t; \varepsilon) \equiv \frac{1}{j_N!}\sum_{j_{N-1}=0}^{j_N}\ldots\sum_{j_2=0}^{j_3}\sum_{j_1=0}^{j_2}\varepsilon_1^{j_1}\varepsilon_2^{j_2-j_1}\ldots\varepsilon_N^{j_N-j_{N-1}} \cdot \mathcal{H}_{j_1, j_2-j_1, \ldots, j_N-j_{N-1}}(\vec{x}, t). \tag{5}$$

It can be readily demonstrated (see Ribera 1981 for the two-parameter case) that, for all $j_N \geqslant 1$, the function $F_{j_N}(\vec{x}, t; \varepsilon)$ is a homogeneous function of j_Nth degree in the variables $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$. In addition, F is a differentiable scalar field defined on $\mathbb{R}^{2n+N}$; the transformed function F^* is also a scalar field of the form

$$F^*(\vec{y}, t; \varepsilon) \equiv \sum_{j_N\geqslant 0}\frac{Y_\varepsilon^{j_N}}{j_N!}F(\vec{y}, t). \tag{6}$$

From Equations (4) and (6) we have

$$\mathcal{H}^*(\vec{y}, t; \varepsilon) = \mathcal{H}_0^*(\vec{y}, t; \varepsilon) + \sum_{j_N\geqslant 1} F^*_{j_N}(\vec{y}, t; \varepsilon), \tag{7}$$

where

$$\mathcal{H}_0^*(\vec{y}, t; \varepsilon) = \sum_{j_N\geqslant 0}\frac{Y_\varepsilon^{j_N}}{j_N!}\mathcal{H}_0(\vec{y}, t) = \mathcal{H}_0(\vec{y}, t) + \sum_{j_N\geqslant 1}\frac{Y_\varepsilon^{j_N}}{j_N!}\mathcal{H}_0(\vec{y}, t). \tag{8}$$

From Equations (5) and (6) we obtain

$$\begin{aligned} F^*_{j_N}(\vec{y}, t; \varepsilon) &\equiv \frac{1}{j_N!}\sum_{j_{N-1}=0}^{j_N}\ldots\sum_{j_2=0}^{j_3}\sum_{j_1=0}^{j_2}\varepsilon_1^{j_1}\varepsilon_2^{j_2-j_1}\ldots\varepsilon_N^{j_N-j_{N-1}} \cdot \mathcal{H}^*_{j_1, j_2-j_1, \ldots, j_N-j_{N-1}}(\vec{y}, t) \\ &= \frac{1}{j_N!}\sum_{j_{N-1}=0}^{j_N}\ldots\sum_{j_2=0}^{j_3}\sum_{j_1=0}^{j_2}\varepsilon_1^{j_1}\varepsilon_2^{j_2-j_1}\ldots\varepsilon_N^{j_N-j_{N-1}} \cdot \sum_{i\geqslant 0}\frac{Y_\varepsilon^i}{i!}\mathcal{H}_{j_1, j_2-j_1, \ldots, j_N-j_{N-1}}(\vec{y}, t) \\ &= \frac{1}{j_N!}\sum_{j_{N-1}=0}^{j_N}\ldots\sum_{j_2=0}^{j_3}\sum_{j_1=0}^{j_2}\varepsilon_1^{j_1}\varepsilon_2^{j_2-j_1}\ldots\varepsilon_N^{j_N-j_{N-1}} \cdot \left[\mathcal{H}_{j_1, j_2-j_1, \ldots, j_N-j_{N-1}}(\vec{y}, t) + \sum_{i\geqslant 1}\frac{Y_\varepsilon^i}{i!}\mathcal{H}_{j_1, j_2-j_1, \ldots, j_N-j_{N-1}}(\vec{y}, t)\right]. \end{aligned} \tag{9}$$

Substituting Equations (8) and (9) into Equation (7) yields Equation (3). □

The so-called remainder function of the nonconservative canonical transformation is

$$\mathcal{R}(\vec{y}, t; \varepsilon) = -\sum_{j_{N-1}\geqslant 1}\frac{Y_\varepsilon^{j_{N-1}-1}}{j_{N-1}!}\mathcal{W}_{\varepsilon t}, \tag{10}$$

where

$$\mathcal{W}_{\varepsilon t} \equiv \frac{\partial \mathcal{W}_\varepsilon}{\partial t}.$$

This function satisfies

$$\nabla_{\vec{y}}\mathcal{R} = -J\,\vec{y}_t.$$

After the transformation, the new canonical system is

$$\dot{\vec{y}} = J\;\nabla_{\vec{y}}\mathcal{K},$$

where the Hamiltonian is given by

$$\mathcal{K}(\vec{y}, t; \varepsilon) = \mathcal{H}^*(\vec{y}, t; \varepsilon) + \mathcal{R}(\vec{y}, t; \varepsilon), \tag{11}$$

which, as we have just seen, can be developed as a power series of the parameters

$$\mathcal{K}(\vec{y}, t; \varepsilon) = \mathcal{K}_0(\vec{y}, t) + \sum_{j_N \geqslant 1} \frac{1}{j_N!} \sum_{j_{N-1}=0}^{j_N} \cdots \sum_{j_2=0}^{j_3} \sum_{j_1=0}^{j_2} \varepsilon_1^{j_1} \varepsilon_2^{j_2-j_1} \times \ldots \varepsilon_N^{j_N-j_{N-1}} \mathcal{K}_{j_1, j_2-j_1, \ldots, j_N-j_{N-1}}(\vec{y}, t).$$

2.2. *Development of the Generating Function as an N-parameter Power Series*

The components $\mathcal{W}^i$ of the generating function $\mathcal{W}_\varepsilon$ can be developed as a power series of the components of the N-dimensional parameter $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$, just as we saw in Equation (2). Then, the Y_ε operator can be written as

$$Y_\varepsilon = \sum_{i=1}^{N} \varepsilon_i \{-; \mathcal{W}^i\} = \sum_{i=1}^{N} \varepsilon_i \left\{-; \sum_{j \geqslant 0} \frac{\varepsilon_i^j}{j!} \mathcal{W}_{j+1}^i \right\}$$
$$= \sum_{i=1}^{N} \varepsilon_i \sum_{j \geqslant 0} \frac{\varepsilon_i^j}{j!} \{-; \mathcal{W}_{j+1}^i\} = \sum_{i=1}^{N} \sum_{j \geqslant 0} \frac{\varepsilon_i^{j+1}}{j!} \{-; \mathcal{W}_{j+1}^i\},$$

or

$$Y_\varepsilon = \sum_{i=1}^{N} \sum_{j \geqslant 0} \frac{\varepsilon_i^{j+1}}{j!} \mathcal{L}_{j+1}^i$$

where we have defined the *Lie operator*

$$\mathcal{L}_j^i = \{-; \mathcal{W}_j^i\}.$$

In general, we have

$$\mathcal{L}_{j_1, \ldots, j_n}^{i_1, \ldots, i_n} = \{\{\ldots\{-; \mathcal{W}_{j_n}^{i_n}\}; \ldots\}; \mathcal{W}_{j_1}^{i_1}\}.$$

We can also prove by induction that

$$Y_\varepsilon^n = \sum_{i_1, \ldots, i_n = 1}^{N} \sum_{j_1, \ldots, j_n \geqslant 0} \frac{\varepsilon_{i_1}^{j_1+1} \varepsilon_{i_2}^{j_2+1} \ldots \varepsilon_{i_n}^{j_n+1}}{j_1! j_2! \ldots j_n!} \mathcal{L}_{j_1+1, j_2+1, \ldots, j_n+1}^{i_1, i_2, \ldots, i_n}.$$

2.3. *Example: Construction of a Four-Parametric Method*

Particularizing for four parameters, we obtain, with $N = 1, 2, 3, 4$ up to second order, a set of relationships. When those are put into Equations (3) and (10), we obtain the expressions for the transformed Hamiltonian and the remainder function, respectively.

2.3.1. *Homological Equations*

Finally, we have to substitute these expressions as well as the new Hamiltonian function as a power series in Equation (11) and to equal the terms of the same order. In this way, we obtain a chain of so-called homological equations:

$$\begin{aligned}
\mathcal{K}_0(\vec{y}, t) &= \mathcal{H}_0(\vec{y}, t), \\
\mathcal{K}_{n000} &= n\left[\{\mathcal{H}_0; \mathcal{W}_n^1\} + \frac{\partial \mathcal{W}_n^1}{\partial t}\right] + \mathcal{A}_{n000}, \\
\mathcal{K}_{0n00} &= n\left[\{\mathcal{H}_0; \mathcal{W}_n^2\} + \frac{\partial \mathcal{W}_n^2}{\partial t}\right] + \mathcal{A}_{0n00}, \\
\mathcal{K}_{00n0} &= n\left[\{\mathcal{H}_0; \mathcal{W}_n^3\} + \frac{\partial \mathcal{W}_n^3}{\partial t}\right] + \mathcal{A}_{00n0}, \\
\mathcal{K}_{000n} &= n\left[\{\mathcal{H}_0; \mathcal{W}_n^4\} + \frac{\partial \mathcal{W}_n^4}{\partial t}\right] + \mathcal{A}_{000n},
\end{aligned} \tag{12}$$

where, up to second order, we define

$$\begin{aligned}
\mathcal{A}_{1000} &= \mathcal{H}_{1000}, & \mathcal{A}_{2000} &= \mathcal{H}_{2000} + \{\mathcal{K}_{1000} + \mathcal{H}_{1000}; \mathcal{W}_1^1\}, \\
\mathcal{A}_{0100} &= \mathcal{H}_{0100}, & \mathcal{A}_{0200} &= \mathcal{H}_{0200} + \{\mathcal{K}_{0100} + \mathcal{H}_{0100}; \mathcal{W}_1^2\}, \\
\mathcal{A}_{0010} &= \mathcal{H}_{0010}, & \mathcal{A}_{0020} &= \mathcal{H}_{0020} + \{\mathcal{K}_{0010} + \mathcal{H}_{0010}; \mathcal{W}_1^3\}, \\
\mathcal{A}_{0001} &= \mathcal{H}_{0001}, & \mathcal{A}_{0002} &= \mathcal{H}_{0002} + \{\mathcal{K}_{0001} + \mathcal{H}_{0001}; \mathcal{W}_1^4\}.
\end{aligned}$$

Thus, the final terms on the right-hand side of Equation (12) are linear combinations of nested Poisson brackets involving the known terms $(\mathcal{H}_{j000})_{1 \leqslant j \leqslant n}$, $(\mathcal{H}_{0j00})_{1 \leqslant j \leqslant n}$, $(\mathcal{H}_{00j0})_{1 \leqslant j \leqslant n}$, $(\mathcal{H}_{000j})_{1 \leqslant j \leqslant n}$, $(\mathcal{W}_j^1)_{1 \leqslant j \leqslant n-1}$, $(\mathcal{W}_j^2)_{1 \leqslant j \leqslant n-1}$, $(\mathcal{W}_j^3)_{1 \leqslant j \leqslant n-1}$, and $(\mathcal{W}_j^4)_{1 \leqslant j \leqslant n-1}$.

Formally, the homological Equations (12) allow us to approach them in two different ways. On the one hand, we may suppose that the generating function $\mathcal{W}$ is known and thus we should determine the new Hamiltonian $\mathcal{K}$ or, on the other hand, we may suppose that we know the new Hamiltonian and therefore we should solve Equation (12) in order to obtain the generating function. The last problem is the most common because the new Hamiltonian is usually chosen in order to exhibit certain properties (see Section 2.3.3).

2.3.2. *Mixed Terms*

The main variation of N-parametric methods with respect to the one-parametric methods is that they give rise to mixed terms in the new Hamiltonian. Indeed, each of these new mixed terms contains the known mixed terms of the old Hamiltonian and linear combinations of nested Poisson brackets involving the nonmixed terms of the old Hamiltonian, similar to the final terms on the right-hand side of Equation (12). Up to second order, the mixed terms of the new Hamiltonian are given by

$$\begin{aligned}
\mathcal{K}_{1100} &= \mathcal{H}_{1100} + \{\mathcal{K}_{1000} + \mathcal{H}_{1000}; \mathcal{W}_1^2\} + \{\mathcal{K}_{0100} + \mathcal{H}_{0100}; \mathcal{W}_1^1\}, \\
\mathcal{K}_{1010} &= \mathcal{H}_{1010} + \{\mathcal{K}_{1000} + \mathcal{H}_{1000}; \mathcal{W}_1^3\} + \{\mathcal{K}_{0010} + \mathcal{H}_{0010}; \mathcal{W}_1^1\}, \\
\mathcal{K}_{1001} &= \mathcal{H}_{1001} + \{\mathcal{K}_{1000} + \mathcal{H}_{1000}; \mathcal{W}_1^4\} + \{\mathcal{K}_{0001} + \mathcal{H}_{0001}; \mathcal{W}_1^1\}, \\
\mathcal{K}_{0110} &= \mathcal{H}_{0110} + \{\mathcal{K}_{0100} + \mathcal{H}_{0100}; \mathcal{W}_1^3\} + \{\mathcal{K}_{0010} + \mathcal{H}_{0010}; \mathcal{W}_1^2\}, \\
\mathcal{K}_{0101} &= \mathcal{H}_{0101} + \{\mathcal{K}_{0100} + \mathcal{H}_{0100}; \mathcal{W}_1^4\} + \{\mathcal{K}_{0001} + \mathcal{H}_{0001}; \mathcal{W}_1^2\}, \\
\mathcal{K}_{0011} &= \mathcal{H}_{0011} + \{\mathcal{K}_{0010} + \mathcal{H}_{0010}; \mathcal{W}_1^4\} + \{\mathcal{K}_{0001} + \mathcal{H}_{0001}; \mathcal{W}_1^3\}.
\end{aligned} \tag{13}$$

We can see that the new nonmixed auxiliary terms $\mathcal{A}$ in (12) and the new mixed terms $\mathcal{K}$ in Equation (13) are formally analogous at n order. Both sets contain the corresponding term of the old Hamiltonian and linear combinations of nested Poisson brackets involving the nonmixed terms of the old Hamiltonian.

2.3.3. *Choice of the New Hamiltonian Terms*

The homological Equations (12) provide recursively the generating function terms $\mathcal{W}_n^1$, $\mathcal{W}_n^2$, $\mathcal{W}_n^3$, and $\mathcal{W}_n^4$, after having conveniently chosen the nonmixed terms $\mathcal{K}_{n000}$, $\mathcal{K}_{0n00}$, $\mathcal{K}_{00n0}$, and $\mathcal{K}_{000n}$ of the new Hamiltonian. This choice must be made so that these terms contain at least all the terms of the functions $\mathcal{A}$ belonging to the kernel of the operator $\mathcal{H}_0 + \frac{\partial \mathcal{W}_n}{\partial t}$. As usual, we adopt the averaging rule

$$\mathcal{K}_{n000} = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{H}_{n000}(\theta)\, d\theta \equiv \langle \mathcal{H}_{n000} \rangle, \tag{14}$$

and do likewise for the remaining three nonmixed terms. Here, θ represents some of the angle variables. After that, the new Hamiltonian $\mathcal{K}$ does not depend on the periodic angle θ, with the result that the system is reduced in one degree of freedom.

Regarding mixed terms, in general there is no freedom in their choice, so that they are completely determined by quantities fixed in previous steps. However, this is less restrictive than it may seem at first. In fact, it does not prevent us from applying the N-parametric method. The only care required is that the new Hamiltonian must be more meticulously handled than in the one-parametric case, especially if there are mixed terms in the old Hamiltonian. This feature was already discussed by Abad & Ribera (1984) in the context of a biparametric method.

We must distinguish two cases: (1) when there are no mixed terms in the old Hamiltonian, the mixed terms of the new Hamiltonian depend on only Lie derivatives of lower-order terms as do the nonmixed terms (see Equation (12)); (2) when there are mixed terms in the old Hamiltonian, the mixed terms of the new Hamiltonian will wholly include them (see Equation (13)), so that if some of these mixed terms contain periodic functions they will appear in the new term as well. In this last case, in order to carry on with the application, we can take advantage of some artifices. For instance, we could resort to applying the method by considering the product of the parameters as another parameter. Otherwise, if we are not able to cast the new Hamiltonian terms in normal form, the N-parametric method would be inapplicable.

3. APPLICATION TO THE PERTURBED GYLDÉN–MEŠČERSKIJ PROBLEM

An interesting problem in binary system dynamics refers to one or both companions losing mass isotropically. This is the well-known Gyldén–Meščerskij problem, the Hamiltonian formulation of which was given by Deprit (1983). The analytical integration of this problem by means of canonical methods with one and two parameters was accomplished by Prieto & Docobo (1997a, 1997b). However, a more complicated situation arises when other perturbations are taken into account, for example the one component's oblateness and/or relativistic effects of the gravitational field.

In this section, we will consider a binary system with mass loss depending on time, the primary's oblateness, and relativistic effects in the first post-Newtonian approximation. The corresponding Hamiltonian is expressed in terms of four small parameters: two of them to quantify the different mass-loss rates of each component, and the other two to measure the primary's oblateness and the relativistic effects, respectively. This Hamiltonian will be analytically integrated in several cases.

3.1. *Analytical Integration by Means of the Four-Parametric Canonical Perturbation Method*

The integration is accomplished by using the four-parametric version of the N-parametric canonical method developed in Section 2. Without loss of generality and with the aim of avoiding cumbersome calculations, we will apply this method at first order.

The old Hamiltonian in Delaunay's variables (Andrade 2007) is given by

$$\mathcal{H} = -\frac{\mu^2}{2L^2} + \frac{\dot\mu}{\mu} Le\sin E + J_2 \frac{\mu^4 R^2}{L^6}\left(\frac{a}{r}\right)^3\left[\frac{1}{4} - \frac{3}{4}\frac{H^2}{G^2} - \frac{3}{4}\left(1-\frac{H^2}{G^2}\right)\cos 2(f+g)\right] - \frac{1}{c^2}\frac{\mu^4}{L^4}\left[\sigma_0 + \left[-\sigma' + \sigma_3\left(\frac{L^2}{G^2}-1\right)\sin^2 f\right]\frac{a}{r} + \sigma''\left(\frac{a}{r}\right)^2\right], \tag{15}$$

in which Delaunay's variables $(L, G, H, \ell, g, (h))$ are a set of canonical variables, with (L, ℓ), (G, g), and (H, h) being pairs of conjugate action-angle variables (L is related to energy, G is the total angular momentum, H is the third component of angular momentum, ℓ is the mean anomaly, g is the argument of the periastron, and h is the angle of the ascending node). Alternatively, $\mu = G(M_1 + M_2)$ is the total stellar mass (M_1 and M_2 are the stellar masses of each component and G is the gravitational constant), r is the distance, f is the true anomaly, E is the eccentric anomaly, e is the eccentricity, and a is the semimajor axis. Further, J_2 and R are the quadrupole moment and the mean equatorial radius of the primary, respectively. As usual, c is the velocity of light.

In addition, the following auxiliary parameters are defined,

$$\sigma' \equiv 4\sigma_0 + \sigma_1$$
$$\sigma'' \equiv 4\sigma_0 + 2\sigma_1 + \sigma_2,$$

with σ_i, for $i = 1, 2, 3$, given by

$$\sigma_0 = \frac{1-3\sigma}{8}, \qquad \sigma_3 = \frac{\sigma}{2},$$
$$\sigma_1 = \frac{3+\sigma}{2}, \qquad \sigma = \frac{\mu_1\mu_2}{(\mu_1+\mu_2)^2}, \tag{16}$$
$$\sigma_2 = -\frac{1}{2}.$$

3.1.1. *Hamiltonian Expansion*

We will consider that the mass-variable law depending on time is the law of Jeans (1924):

$$\dot\mu = -\alpha\mu^n,$$

where μ is the stellar mass, α is a positive small parameter, and $1.4 \lesssim n \lesssim 4.4$. We take this expression for each component of the binary system, so that we have two small parameters α_1 and α_2.

Then the Hamiltonian problem (15) depends on the four small parameters α_1, α_2 (related to the mass-loss rates), J_2 (dynamical form factor), and c^{-2}. Therefore, we can consider its expansion as a series of these small parameters with the aim to apply the four-parametric canonical method.

Up to first order, the series expansion of the Hamiltonian is given by

$$\mathcal{H} = \mathcal{H}_0 + \alpha_1\,\mathcal{H}_{1000} + \alpha_2\,\mathcal{H}_{0100} + J_2\,\mathcal{H}_{0010} + \frac{1}{c^2}\,\mathcal{H}_{0001},$$

where

$$\mathcal{H}_0 = -\frac{\mu_0^2}{2\,L^2},$$

$$\mathcal{H}_{1000} = \frac{\mu_{10}^{n_1}\mu_0}{L^2}\,(t-t_0) - \frac{\mu_{10}^{n_1}}{\mu_0}\,L\,e\sin E,$$

$$\mathcal{H}_{0100} = \frac{\mu_{20}^{n_2}\mu_0}{L^2}\,(t-t_0) - \frac{\mu_{20}^{n_2}}{\mu_0}\,L\,e\sin E,$$

$$\mathcal{H}_{0010} = \frac{\mu_0^4\,R^2}{4\,G^6}\,(1+e\cos f)^3\left[1-3\,\frac{H^2}{G^2} - 3\left(1-\frac{H^2}{G^2}\right)\times\ \cos 2(f+g)\right],$$

$$\mathcal{H}_{0001} = -\frac{\mu_0^4}{L^4}\left[\sigma_0 + \left[-\sigma' + \sigma_3\left(\frac{L^2}{G^2}-1\right)\sin^2 f\right]\frac{L^2}{G^2}\times\ (1+e\cos f) + \sigma''\,\frac{L^4}{G^4}\,(1+e\cos f)^2\right],$$

with $\mu_0 = \mu_{10} + \mu_{20}$, μ_{i0} being the mass i at t_0.

3.1.2. Elimination of ℓ by Means of the Four-Parametric Method

We look for an infinitesimal canonical transformation of the type

$$(L, G, H, \ell, g, h) \leftrightarrow (L^*, G^*, H^*, \ell^*, g^*, h^*),$$

so that the new Hamiltonian

$$\mathcal{H}^* \equiv \mathcal{H}^*(L^*, G^*, H^*;\ -, g^*, -)$$

is independent of some angular variable and, therefore, more easily integrable. In this case, the short-period terms depending on ℓ will be eliminated. Apart from this, the long-period terms depending on g^* will also disappear due to the first-order approximation.

The homological Equations (12) to the first order allow us to obtain successively the corresponding terms $\mathcal{W}_1^1$, $\mathcal{W}_1^2$, $\mathcal{W}_1^3$, and $\mathcal{W}_1^4$, after having appropriately chosen the terms $\mathcal{H}^*_{1000}$, $\mathcal{H}^*_{0100}$, $\mathcal{H}^*_{0010}$, and $\mathcal{H}^*_{0001}$ of the new Hamiltonian.

In this case, since the partial derivatives of the generating function components are time independent, we have

$$\frac{\partial \mathcal{W}_1^j}{\partial t} = 0, \qquad j = 1, 2, 3, 4.$$

At zeroth order

$$\mathcal{H}_0^* = \mathcal{H}_0.$$

At first order, after averaging on the variable ℓ^* we obtain

$$\begin{aligned}
\mathcal{H}^*_{1000} &= \frac{\mu_{10}^{n_1}\mu_0}{L^{*2}}\,(t-t_0),\\
\mathcal{H}^*_{0100} &= \frac{\mu_{20}^{n_2}\mu_0}{L^{*2}}\,(t-t_0),\\
\mathcal{H}^*_{0010} &= \frac{\mu_0^4\,R^2}{4\,L^{*3}G^{*3}}\left[1-3\,\frac{H^{*2}}{G^{*2}}\right],\\
\mathcal{H}^*_{0001} &= -\frac{\mu_0^4}{L^{*4}}\left[-3\,\sigma_0 - \sigma_1 + \sigma_3\left(\frac{L^*}{G^*}-1\right) + (4\,\sigma_0 + 2\,\sigma_1 + \sigma_2)\,\frac{L^*}{G^*}\right].
\end{aligned} \tag{17}$$

The generating function of this Lie transformation is

$$\mathcal{W}_1^{(\alpha_1,\alpha_2,J_2,c^{-2})} = \alpha_1\,\mathcal{W}_1^1 + \alpha_2\,\mathcal{W}_1^2 + J_2\,\mathcal{W}_1^3 + c^{-2}\,\mathcal{W}_1^4,$$

the components of which are calculated from Equations (12) and (17).

To first order, the Lie transformation that gives the change of variables can be written as

$$\begin{aligned}
L &= L^* - \frac{\partial \mathcal{W}_1}{\partial \ell^*}, & \ell &= \ell^* + \frac{\partial \mathcal{W}_1}{\partial L^*},\\
G &= G^* - \frac{\partial \mathcal{W}_1}{\partial g^*}, & g &= g^* + \frac{\partial \mathcal{W}_1}{\partial G^*},\\
H &= H^* - \frac{\partial \mathcal{W}_1}{\partial h^*}, & h &= h^* + \frac{\partial \mathcal{W}_1}{\partial H^*},
\end{aligned} \tag{18}$$

where the partial derivatives can now be calculated from the known components of the generating function.

Because of axial symmetry, the generating function is independent of h^*, so that from Equation (18) it is inferred that

$$H = H^*.$$

However, since the new Hamiltonian $\mathcal{H}^*$ is independent of the angular variables ℓ^*, g^*, and h^*, the canonical equations of Hamilton can be integrated so that

$$\begin{aligned}
\frac{dL^*}{dt} &= -\frac{\partial \mathcal{H}^*}{\partial \ell^*} = 0 \Rightarrow L^* = cte,\\
\frac{dG^*}{dt} &= -\frac{\partial \mathcal{H}^*}{\partial g^*} = 0 \Rightarrow G^* = cte,\\
\frac{dH^*}{dt} &= -\frac{\partial \mathcal{H}^*}{\partial h^*} = 0 \Rightarrow H^* = cte.
\end{aligned}$$

The remaining canonical equations are integrated at first order. The equation for the mean anomaly reads

$$\begin{aligned}
\frac{d\ell^*}{dt} = \frac{\partial \mathcal{H}^*}{\partial L^*} = &+\frac{\mu_0^2}{L^{*3}} - \alpha_1\,\frac{2\mu_{10}^{n_1}\mu_0}{L^{*3}}\,(t-t_0) - \alpha_2\,\frac{2\mu_{20}^{n_2}\mu_0}{L^{*3}}\\
&\times (t-t_0) - J_2\,\frac{3\mu_0^4 R^2}{4L^{*4}G^{*3}}\left[1-3\,\frac{H^{*2}}{G^{*2}}\right]\\
&+ c^{-2}\,\frac{\mu_0^4}{L^{*5}}\left[3(\sigma''+\sigma_3)\frac{L^*}{G^*} + 4(\sigma_0 - \sigma' - \sigma_3)\right].
\end{aligned}$$

From this last equation we obtain

$$\ell^* = \ell_0^* + \ell_1^* t + \ell_2^* t^2,$$

where

$$\ell_1^* = +\frac{\mu_0^2}{L^{*3}} + \alpha_1 \frac{2\mu_{10}^{n_1}\mu_0}{L^{*3}} t_0 + \alpha_2 \frac{2\mu_{20}^{n_2}\mu_0}{L^{*3}} t_0 - J_2 \frac{3\mu_0^4 R^2}{4L^{*4}G^{*3}} \times \left[1 - 3\frac{H^{*2}}{G^{*2}}\right] + c^{-2}\frac{\mu_0^4}{L^{*5}}\left[3(\sigma'' + \sigma_3)\frac{L^*}{G^*} + 4(\sigma_0 - \sigma' - \sigma_3)\right],$$

$$\ell_2^* = -\alpha_1 \frac{\mu_{10}^{n_1}\mu_0}{L^{*3}} - \alpha_2 \frac{\mu_{20}^{n_2}\mu_0}{L^{*3}},$$

with ℓ_0^* being a constant of integration.

Similarly, for the argument of the periastron

$$\frac{dg^*}{dt} = \frac{\partial \mathcal{H}^*}{\partial G^*} = -J_2 \frac{3\mu_0^4 R^2}{4L^{*3}G^{*4}}\left[1 - 5\frac{H^{*2}}{G^{*2}}\right] + c^{-2}\frac{\mu_0^4}{L^{*3}G^{*2}}(\sigma'' + \sigma_3),$$

and by integrating

$$g^* = g_0^* + g_1^* t$$

where

$$g_1^* = -J_2 \frac{3\mu_0^4 R^2}{4L^{*3}G^{*4}}\left[1 - 5\frac{H^{*2}}{G^{*2}}\right] + c^{-2}\frac{\mu_0^4}{L^{*4}G^{*2}}(\sigma'' + \sigma_3),$$

with g_0^* being a constant of integration.

Finally, for the angle of the node

$$\frac{dh^*}{dt} = \frac{\partial \mathcal{H}^*}{\partial H^*} = -J_2 \frac{3\mu_0^4 R^2}{2L^{*3}G^{*4}}\frac{H^*}{G^*},$$

so that

$$h^* = h_0^* + h_1^* t,$$

where

$$h_1^* = -J_2 \frac{3\mu_0^4 R^2}{2L^{*3}G^{*4}}\frac{H^*}{G^*},$$

with h_0^* being a constant of integration.

3.1.3. Ephemerids Calculation

We take a set of initial values of the orbital elements

$$\left(a_0^*, e_0^*, i_0^*, T_0^*, \omega_0^*, \Omega_0^*\right)$$

at t_0 with the initial mass μ_0.

The corresponding Delaunay variables are obtained:

$$L^*{}_0 = \sqrt{\mu_0 a_0^*}, \qquad \ell_0^* = \sqrt{\frac{\mu_0}{a^{*3}_0}}(t_0 - T_0),$$

$$G^*{}_0 = L^*{}_0\sqrt{1 - e^{*2}_0}, \qquad g_0^* = \omega_0^*,$$

$$H^*{}_0 = G^*{}_0 \cos i^*, \qquad h_0^* = \Omega_0^*.$$

The constants of the motion L^*, G^*, H^*, ℓ_0^*, g_0^* and h_0^* are calculated according to

$$L^* = L^*{}_0 + \left(\frac{\partial \mathcal{W}}{\partial \ell^*}\right)_0, \qquad \ell_0^* = \ell_0^* - \left(\frac{\partial \mathcal{W}}{\partial L^*}\right)_0,$$

$$G^* = G^*{}_0 + \left(\frac{\partial \mathcal{W}}{\partial g^*}\right)_0, \qquad g_0^* = g_0^* - \left(\frac{\partial \mathcal{W}}{\partial G^*}\right)_0,$$

$$H^* = H^*{}_0 + \left(\frac{\partial \mathcal{W}}{\partial h^*}\right)_0 = H_0^*, \qquad h_0^* = h_0^* - \left(\frac{\partial \mathcal{W}}{\partial H^*}\right)_0,$$

$$\ell^* = \ell_0^* + \ell_1^* t + \ell_2^* t^2,$$
$$g^* = g_0^* + g_1^* t,$$
$$h^* = h_0^* + h_1^* t.$$

The orbital elements are obtained at any instant t

$$a = \frac{L^2}{\mu}, \qquad T = t - \ell\frac{L^3}{\mu^2},$$

$$e = \sqrt{1 - \frac{G^2}{L^2}}, \qquad \omega = g,$$

$$i = \arccos\left(\frac{H}{G}\right), \qquad \Omega = h.$$

It is customary to substitute the time of passage from the periastron by the eccentric anomaly or by the true anomaly. In fact, we will use the latter in our calculations.

When one considers the second order, long-period terms depending on the argument of the periastron according to $\cos 2g^*$ appear in the mixed term $J_2 c^{-2}$ of the new Hamiltonian. To eliminate these it should be necessary to apply the method again, now in its two-parametric version considering an infinitesimal canonical transformation of the form

$$(L^*, G^*, H^*, \ell^*, g^*, h^*) \leftrightarrow (L^{**}, G^{**}, H^{**}, \ell^{**}, g^{**}, h^{**}).$$

Therefore, the new Hamiltonian

$$\mathcal{H}^{**} \equiv \mathcal{H}^{**}(L^{**}, G^{**}, H^{**};\ -, -, -)$$

will be independent of the angular variable g^*, so that only secular terms remain in the new Hamiltonian.

3.2. Practical Example

The implementation of this four-parametric version of the method was accomplished by using a code programmed in the *Mathematica* package. The program was tested with well-known examples of the influence of the Earth's oblateness on artificial satellite orbits (Soffel et al. 1988) and the relativistic effects produced by the solar gravitational field on the motion of Mercury (Richardson & Kelly 1988) at first order. In this way, the results obtained by other authors in both periodic variations of eccentricity and semimajor axis, and in secular variations of the argument of the periastron and the angle of the node, were matched.

In order to demonstrate the power of this method, in what follows we integrate the motion of a binary system, the orbital elements and physical parameters of which are listed in

Table 1
Orbital Elements and Parameters of the System

Orbital elements		Parameters	
T	2000	m_1 ($M_\odot$)	15
e	0.2	m_2 ($M_\odot$)	1.0
a (AU)	0.5	α_1	10^{-4}
i (°)	50	α_2	10^{-6}
Ω (°)	40	n_1	1.5
ω (°)	20	n_2	3.0
		R_1 ($R_\odot$)	5.0
		J_2	10^{-5}

Table 2
Secular Variations of the Orbital Elements at the End of 1 Year and Perturbations That Produce Them

Values at the end of 1 year		Phenomenon
ΔT	$-3.68 \cdot 10^{-4}$ yr	Mass loss
Δa	$+1.83 \cdot 10^{-4}$ AU	Mass loss
$\Delta\Omega$	$-0''.34$	Oblate shape
$\Delta\omega$	$+0''.28$ (1.6%)	Oblate shape
	$+17''.63$ (98.4%)	Relativistic effects

Table 3
Speed and Accuracy Comparisons Between the First-order Analytical Method (This Paper) and the Numerical Method at the End of 1 Year

	Analytical method (N-parametric canonical)	Numerical method (Implicit Runge–Kutta)
	$\Delta t = 10^{-3}$ yr	
CT (s)	45	152
ΔE	$+3.82625 \cdot 10^{-7}$	$+3.72810 \cdot 10^{-7}$
ΔG	$-6.48871 \cdot 10^{-9}$	$-2.40164 \cdot 10^{-8}$
ΔT (yr)	$-3.68546 \cdot 10^{-4}$	$-3.66532 \cdot 10^{-4}$
Δa (AU)	$+1.83464 \cdot 10^{-4}$	$+1.82938 \cdot 10^{-4}$
$\Delta\Omega$ ($''$)	-0.337921	-0.333744
$\Delta\omega$ ($''$)	$+25.8237$	$+24.6070$
	$\Delta t = 10^{-4}$ yr	
CT (s)	3509	1688
ΔE	$+3.82624 \cdot 10^{-7}$	$+3.72810 \cdot 10^{-7}$
ΔG	$-6.48871 \cdot 10^{-9}$	$-2.40164 \cdot 10^{-8}$
ΔT (yr)	$-3.67888 \cdot 10^{-4}$	$-3.67588 \cdot 10^{-4}$
Δa (AU)	$+1.83299 \cdot 10^{-4}$	$+1.83191 \cdot 10^{-4}$
$\Delta\Omega$ ($''$)	-0.337921	-0.333991
$\Delta\omega$ ($''$)	$+25.8237$	$+25.0613$

Table 1. In the first column we give a set of initial orbital elements: semimajor axis (in astronomical units), eccentricity, inclination (in degrees), time of passage from the periastron (in years), argument of the periastron (in degrees), and the angle of the node (in degrees). They are followed by the physical parameters of the system in the next column: masses (in solar masses), mass-loss rate parameters, exponents in Jeans' law, primary stellar radius (in solar radii), and quadrupole moment. The value of the gravitational constant was taken as $G = 4\pi^2$ AU^3 $M_\odot^{-1}$ yr^{-2}.

3.2.1. Periodic Terms and Secular Variations at First Order

After operations carried out in Subsection 3.1, a new first-order Hamiltonian free of short-period variations has been obtained, which can be considered an average (Ferraz-Mello 2007). Because of the first-order integration, some long-period terms, those due to the primary's oblateness, remain at second order. In order to properly understand the results, we have separately analyzed the contribution of each perturbation under consideration to the observed variations.

Mass loss causes short-period variations of the eccentricity and of the argument of the periastron. It has no effect on the inclination or the angle of the node, although it is responsible for the secular increments of the semimajor axis and time of passage from the periastron.

In addition, when the relativistic effects are considered in the first post-Newtonian approximation, short-period variations of all the orbital elements appear, except for the inclination and the angle of the node, which remain unaltered. However, the most remarkable variation is the secular motion of the argument of the periastron.

With regard to the primary's oblateness, despite the first-order approximation, we obtain one of most remarkable variations, the secular retrograde motion in the angle of the node. Besides this, the secular motion of the argument of the periastron is also obtained. However, we must mention that an exhaustive study of this perturbation would require a second application of the method to eliminate long-period terms depending on the other angle variable g^* from the second-order Hamiltonian. The accomplishment of that procedure is, however, beyond the scope of this paper, where we focus essentially upon the derivation of a multiparametric theory based on Lie transforms.

The secular variations that arise in this problem at the end of 1 yr are summarized in Table 2, where one can see what perturbation is responsible for each variation. Thus, ΔT and Δa are entirely governed by mass loss; $\Delta\Omega$ is utterly caused by the primary's oblateness, whereas in this particular case $\Delta\omega$ is divided up in the following way: 1.6% on account of the primary's oblateness, while the remaining 98.4% is produced by relativistic effects.

From the equations that describe the motion of the system, it is inferred that, except for the semimajor axis and the time of passage from the periastron, which always vary secularly due to the mass loss, the increase direction for the argument of the periastron and the angle of the node depends on orbital inclination:

$$\Delta\omega \sim 5\cos^2 i - 1$$
$$\Delta\Omega \sim \cos i.$$

By considering the scenario in that the primary shows an oblate shape, a critical inclination $i_{\text{critical}} = 1/\sqrt{5} \simeq 63^\circ\!.4$ exists for which the argument of the periastron does not show secular variations. For higher values this orbital element will begin to retrograde. In fact, when the inclination is 90° the retrograde motion for the argument of the periastron reaches the maximum, while the angle of the node remains constant.

3.2.2. Comparison with a Numerical Method

In order to compare both speed and accuracy, we performed several integrations of the Hamiltonian given in Equation (15) with parameters shown in Table 1 by alternately using the N-parametric canonical method (this paper; thereafter, NpCM) and an implicit Runge–Kutta method (thereafter, IRKM). In both cases integrations spanned 1 yr and were carried out taking into account integration step sizes $\Delta t = 10^{-3}$ yr and $\Delta t = 10^{-4}$ yr. All operations were performed in a 2.66 GHz *Pentium* IV processor with 3 Gb of RAM.

Comparison results are summarized in Table 3, where the first column lists the evaluated parameters and the last two

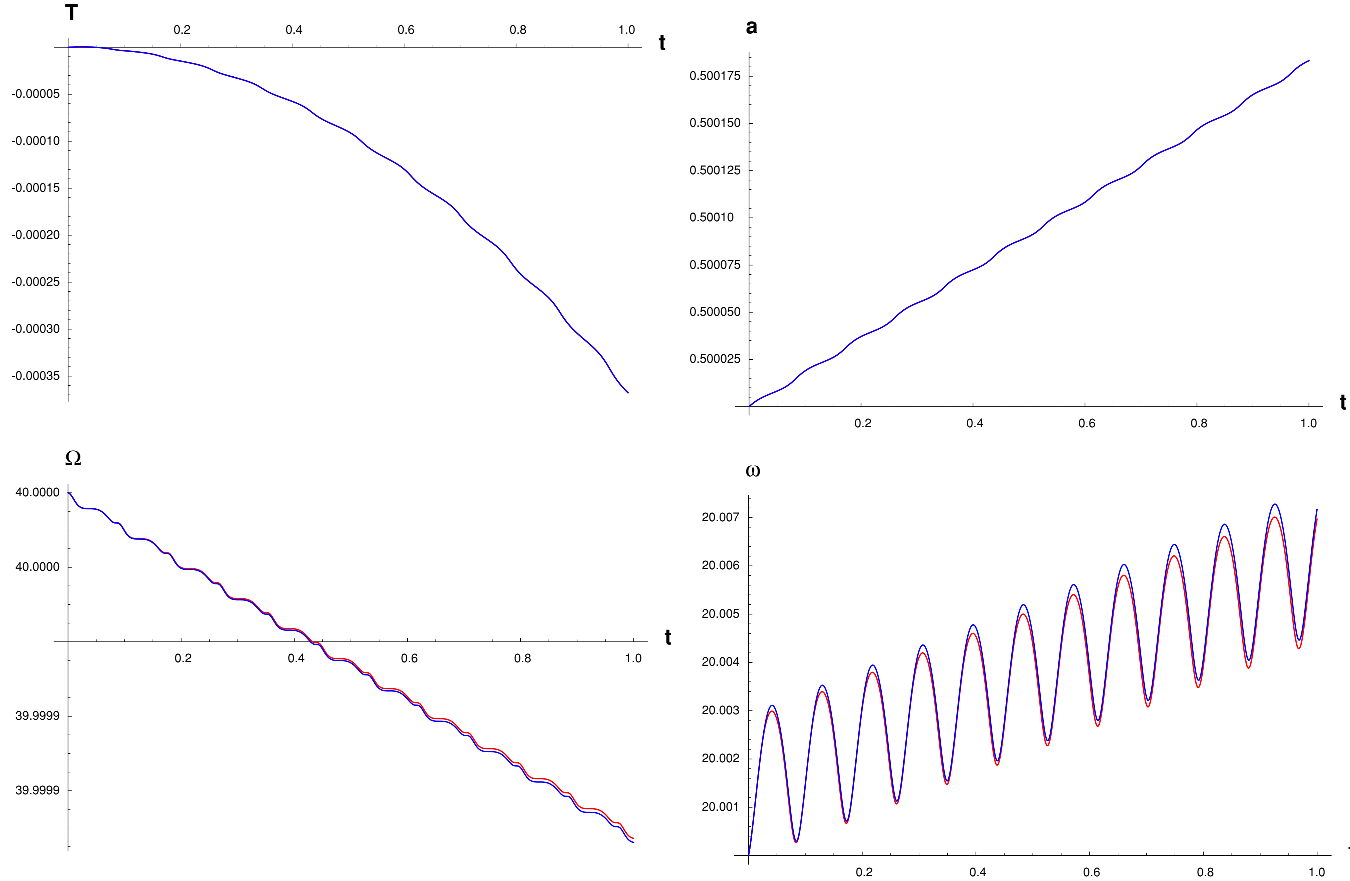

Figure 1. Secular variations on the orbital elements for the system spanning 1 year (NpCM in blue and IRKM in red).

list the corresponding results for the NpCM and the IRKM, respectively. The first parameter gives the computing time (CT, in seconds), whereas the next two rows give energy and angular momentum conservation, respectively. The last four rows list secular variations of time of passage from the periastron (in years), the semimajor axis (in astronomical units), the angle of the node (in arcseconds), and the argument of the periastron (in arcseconds). On the other hand, Table 3 is divided into two parts as well. The upper part shows results obtained with a integration step size $\Delta t = 10^{-3}$ yr, whereas the lower part shows those obtained with a smaller integration step size $\Delta t = 10^{-4}$ yr. The time-dependent evolution of the orbital elements undergoing secular variations is shown in Figure 1, where blue and red lines refer to NpCM and IRKM, respectively.

From an analysis on Table 3, we conclude that NpCM is very fast with the largest integration step size ($\Delta t = 10^{-3}$ yr), exactly 3.4 times faster than IRKM. In contrast, IRKM is 2.1 times faster than NpCM when $\Delta t = 10^{-4}$ yr. From the point of view of accuracy we want to emphasize that NpCM gives similar results independently of the integration step sizes. In fact, we can see that, for instance, $\Delta\Omega$ and $\Delta\omega$ converge to the same value for NpCM in both cases, whereas they are relatively different for IRKM. Similar results are also obtained by using the IRKM with the small integration step size, but at the expense of consuming much more time. However, both integrators conserve energy to better than 10^{-7}; moreover, NpCM conserves angular momentum to better than 10^{-9}, one order better than IRKM. In short, NpCM allows us to obtain accurate results with relatively large integration step sizes or, in other words, with short computing times.

4. SUMMARY

Lie transforms have proved to be a powerful tool to solve nonlinear differential equations. In particular, canonical methods of perturbations defined from them are very suitable to treat Hamiltonian systems depending on small parameters.

In accordance with this, most meaningful achievements of the N-parametric (analytical) canonical method presented in this paper can be summarized as follows.

1. This method allows us to analytically solve problems involving an arbitrary number of superposed perturbations as a whole.
2. We have derived general expressions depending on N small parameters that can be easily handled in order to obtain a set of $N(+1)$ homological equations, the solution of which allows us to obtain the generating function of the transformation up to a certain order.
3. In addition, a set of mixed terms is obtained. This is the most important difference in regard to one-parameter methods, which do not exhibit mixed terms. Before anything else, it is essential to verify whether the old Hamiltonian exhibits mixed terms including periodic functions or not, since they would appear ineluctably as mixed terms of the new Hamiltonian.
4. In order to solve homological equations, it is customary to average non-mixed terms of the Hamiltonian over certain angle variables—in the same way as it is accomplished for one-parameter methods. Thus, the new Hamiltonian does not depend on the periodic angles.

5. Lastly, Lie series structure is very appropriate to construct programmable schemes by using algebraic manipulators.

From another point of view, with the aim to evaluate this method, we have considered a stellar binary system undergoing altogether four perturbations: mass loss in both components, oblateness of the primary and relativistic effects of the gravitational field. The following outlines the main conclusions concerning the application of the method to this perturbed Gyldén–Meščerskij problem.

1. A four-parametric version of the N-parametric canonical method developed in Section 2 was applied at first order, without loss of generality, to the Hamiltonian of this system in order to eliminate short-period terms depending on the mean anomaly.
2. Therefore, the new Hamiltonian exhibits only secular terms. Despite this, a long-period part due to the oblateness remains at second order. This last also could be eliminated by applying a second transformation to the second-order Hamiltonian, but such a goal is, however, beyond the scope of this paper.
3. The most remarkable results obtained with this method regarding the motion of this perturbed binary system are the secular variations of the time of passage from the periastron, the semimajor axis, the angle of the node, and the argument of the periastron.
4. Both speed and accuracy tests have been performed by comparing first-order analytical and numerical integrations. These integrations have been carried out taking into account the perturbations altogether. From them, we conclude that the N-parametric canonical method allows us to obtain accurate results with relatively large integration step sizes, which is the same as with short computing times.

Finally, we want to emphasize that the applicability of the N-parametric canonical method presented in this paper is not circumscribed to the multiple stellar systems field, so that it can be used to integrate Hamiltonian systems depending on N parameters not only in astronomy but also in other branches of mathematics and physics.

The author wishes to thank J. A. Docobo for his encouragement and useful discussions. The suggestions and comments of A. Abad are also appreciated. Finally, the author extends his gratitude to an anonymous referee for a careful reading of the manuscript and many highly constructive suggestions. This paper was financed by research project AYA 2004-07003 of the Spanish Ministerio de Educación y Ciencia.

REFERENCES

Abad, A. J., & Ribera, J. 1984, in Publicaciones del Seminario Matemático García de Galdeano (Zaragoza, Spain: Universidad de Zaragoza), II(2)-3, 1
Ahmed, M. K. M. 1993, Earth Moon Planets, 61, 21
Andrade, M. 2002, in Métodos de dinámica orbital y rotacional, ed. S. Ferrer et al. (Murcia, Spain: Prensas Universitarias), 113
Andrade, M. 2007, PhD thesis, Universidade de Santiago de Compostela, Galiza, Spain
Campbell, J. A., & Jefferys, W. A. 1970, Celest. Mech., 2, 467
Deprit, A. 1969, Celest. Mech., 1, 12
Deprit, A. 1983, Celest. Mech., 31, 1
Docobo, J. A., & Andrade, M. 2002, in Métodos de dinámica orbital y rotacional, ed. S. Ferrer et al. (Murcia, Spain: Prensas Universitarias), 121
Ferraz-Mello, S. 2007, in Canonical Perturbation Theories: Degenerate Systems and Resonance (New York: Springer), 158–159
Henrard, J. 1970, Celest. Mech., 3, 107
Henrard, J., & Roels, J. 1974, Celest. Mech., 10, 497
Hori, G. I. 1966, PASJ, 18, 287
Jeans, J. H. 1924, MNRAS, 85, 2
Kamel, A. 1970, Celest. Mech., 3, 90
Prieto, C., & Docobo, J. A. 1997a, A&A, 318, 657
Prieto, C., & Docobo, J. A. 1997b, Celest. Mech. Dyn. Astron., 68, 53
Ribera, J. 1981, PhD thesis, Universidad de Zaragoza, Spain
Richardson, D. L., & Kelly, T. J. 1988, Celest. Mech., 43, 193
Soffel, M., Wirrer, R., Schastok, J., Ruder, H., & Schneider, M. 1988, Celest. Mech., 42, 81
Varadi, F. 1985, Celest. Mech., 36, 133